Divine Proportions Solutions
Chapter 7: Triple spread formula
Exercises 7.13 to 7.15

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Exercise 7.13 (p.100)

Show that if $\{a, b, x\}$ and $\{b, c, x\}$ are spread triples with $a \neq c$ and $b \neq \frac{1}{2}$ then

$$x = \frac{a + c - 2b}{2(1 - 2b)}.$$

Solution

If $\{a, b, x\}$ and $\{b, c, x\}$ are spread triples, then using Exercise 7.6 (p.91) we have the pair of quadratic equations

$$\begin{cases} (x - (a + b - 2ab))^2 = 4ab(1 - a)(1 - b) \\ (x - (b + c - 2bc))^2 = 4bc(1 - b)(1 - c) \end{cases}$$

in x. Then, the Two spread triples theorem (p.98) tells us that $\{a, b, b, c\}$ is a spread quadruple and thus as long as

$$a - 2ab = a(1 - 2b) \neq c - 2bc = c(1 - 2b),$$

which holds as long as $a \neq c$ and $b \neq \frac{1}{2}$, we then have

$$x = \frac{(a - b)^2 - (b - c)^2}{2(a + b - b - c - 2ab + 2bc)} = \frac{a + c - 2b}{2(1 - 2b)}$$

as required.

Exercise 7.14 (p.100)

Suppose l_1, l_2, l_3 and l_4 are non-null lines with $c_{ij} \equiv c(l_i, l_j)$ for all $i, j = 1, 2, 3, 4$. Then, show that

$$\begin{aligned} c_{13} &= \frac{(c_{12} - c_{23})^2 - (c_{34} - c_{14})^2}{2(c_{12} + c_{23} - c_{34} - c_{14} - 2c_{12}c_{23} + 2c_{34}c_{14})} \\ c_{24} &= \frac{(c_{23} - c_{34})^2 - (c_{12} - c_{14})^2}{2(c_{23} + c_{34} - c_{12} - c_{14} - 2c_{23}c_{34} + 2c_{12}c_{14})}. \end{aligned}$$

Solution

We start with the fact that $\{c_{12}, c_{23}, c_{13}\}$, $\{c_{13}, c_{34}, c_{14}\}$, $\{c_{12}, c_{24}, c_{14}\}$ and $\{c_{23}, c_{34}, c_{24}\}$ are all spread triples. Then, using Exercise 7.6 we have the two pairs of quadratic equations

$$\begin{cases} (c_{13} - (c_{12} + c_{23} - 2c_{12}c_{23}))^2 = 4c_{12}c_{23}(1 - c_{12})(1 - c_{23}) \\ (c_{13} - (c_{14} + c_{34} - 2c_{14}c_{34}))^2 = 4c_{14}c_{34}(1 - c_{14})(1 - c_{34}) \end{cases}$$

and

$$\begin{cases} (c_{24} - (c_{12} + c_{14} - 2c_{12}c_{14}))^2 = 4c_{12}c_{14}(1 - c_{12})(1 - c_{14}) \\ (c_{24} - (c_{23} + c_{34} - 2c_{23}c_{34}))^2 = 4c_{23}c_{34}(1 - c_{23})(1 - c_{34}) \end{cases}.$$

Given the fact that $\{c_{12}, c_{23}, c_{34}, c_{14}\}$ is a spread quadruple, the Two spread triples theorem (p. 98) ensures that these pairs of quadratic equations are compatible, and that

$$\begin{aligned} c_{13} &= \frac{(c_{12} + c_{23} - 2c_{12}c_{23}) + (c_{14} + c_{34} - 2c_{14}c_{34})}{2} \\ &\quad - \frac{4c_{12}c_{23}(1 - c_{12})(1 - c_{23}) - 4c_{14}c_{34}(1 - c_{14})(1 - c_{34})}{2((c_{12} + c_{23} - 2c_{12}c_{23}) - (c_{14} + c_{34} - 2c_{14}c_{34}))} \\ &= \frac{(c_{12} - c_{23})^2 - (c_{34} - c_{14})^2}{2(c_{12} + c_{23} - c_{34} - c_{14} - 2c_{12}c_{23} + 2c_{34}c_{14})} \end{aligned}$$

and

$$\begin{aligned} c_{24} &= \frac{(c_{12} + c_{14} - 2c_{12}c_{14}) + (c_{23} + c_{34} - 2c_{23}c_{34})}{2} \\ &\quad - \frac{4c_{12}c_{14}(1 - c_{12})(1 - c_{14}) - 4c_{23}c_{34}(1 - c_{23})(1 - c_{34})}{2((c_{12} + c_{14} - 2c_{12}c_{14}) - (c_{23} + c_{34} - 2c_{23}c_{34}))} \\ &= \frac{(c_{23} - c_{34})^2 - (c_{12} - c_{14})^2}{2(c_{23} + c_{34} - c_{12} - c_{14} - 2c_{23}c_{34} + 2c_{12}c_{14})} \end{aligned}$$

as required.

Exercise 7.15 (p.100)

Is there a Quintuple spread formula? Generalise.

Solution

This is left as an open problem, which is still currently unsolved.