Divine Proportions Solutions
Chapter 7: Triple spread formula
Exercises 7.7 to 7.12

Dr. Gennady Arshad Notowidigdo, PhD

Exercise 7.7 (p.92)

Show that as a quadratic equation in c_3, the Triple cross formula in normal form is

$$(c_3 + c_1 + c_2 - 2c_1c_2 - 1)^2 = 4c_1c_2(1 - c_1)(1 - c_2)$$

or also incorporating the spreads,

$$(c_3 - (c_1c_2 + s_1s_2))^2 = 4c_1c_2s_1s_2.$$

Solution

Given the Triple cross formula (p.92)

$$(c_1 + c_2 + c_3 - 1)^2 = 4c_1c_2c_3,$$

we rearrange it as a quadratic equation in c_3 to get

$$c_3^2 + 2(c_1 + c_2 - 2c_1c_2 - 1)c_3 + (c_1 + c_2 - 1)^2 = 0.$$

Complete the square to obtain

$$\begin{aligned}(c_3 + c_1 + c_2 - 2c_1c_2 - 1)^2 &= (c_1 + c_2 - 2c_1c_2 - 1)^2 - (c_1 + c_2 - 1)^2 \\ &= 4c_1c_2(1 - c_1)(1 - c_2)\end{aligned}$$

in normal form, as required. Incorporate the spreads by setting $c_1 \equiv 1 - s_1$ and $c_2 \equiv 1 - s_2$, so that our equation now becomes

$$\begin{aligned}4c_1c_2s_1s_2 &= (c_3 + 1 - s_1 + 1 - s_2 - (1 - s_1)(1 - s_2) - 1 - c_1c_2)^2 \\ &= (c_3 - c_1c_2 - s_1s_2)^2.\end{aligned}$$

The required result follows.

Exercise 7.8 (p.92)

Following Exercises 5.7 (p.64) and 7.1 (p.90), show that

$$(c_1+c_2+c_3-1)^2-4c_1c_2c_3=-\det\begin{pmatrix}1 & c_1 & c_2 & 1\\ c_1 & 1 & c_3 & 1\\ c_2 & c_3 & 1 & 1\\ 1 & 1 & 1 & 2\end{pmatrix}.$$

Solution

We simply start with the right-hand side and see that

$$\begin{aligned}-\det\begin{pmatrix}1 & c_1 & c_2 & 1\\ c_1 & 1 & c_3 & 1\\ c_2 & c_3 & 1 & 1\\ 1 & 1 & 1 & 2\end{pmatrix} &= \begin{pmatrix} c_1^2+c_2^2+c_3^2+2c_1c_2+2c_1c_3+2c_2c_3 \\ -2c_1-2c_2-2c_3+1\end{pmatrix}-4c_1c_2c_3\\ &= (c_1+c_2+c_3-1)^2-4c_1c_2c_3\end{aligned}$$

as required.

Exercise 7.9 (p.92)

Please note that the statement of the question is incorrect (see the errata provided with the book by Wildberger).

Exercise 7.10 (p.93)

Rewrite the Triple twist formula as a quadratic equation in t_3 in normal form.

Solution

Given the Triple twist formula (p.93)

$$(t_1+t_2+t_3-t_1t_2t_3)^2=4\left(t_1t_2+t_1t_3+t_2t_3+2t_1t_2t_3\right),$$

we rearrange and expand to obtain the equation

$$\begin{aligned}0 &= t_1^2+t_2^2+t_3^2+t_1^2t_2^2t_3^2-2t_1^2t_2t_3-2t_1t_2^2t_3-2t_1t_2t_3^2-2t_1t_2-2t_1t_3-2t_2t_3-8t_1t_2t_3\\ &= \left(t_1^2t_2^2-2t_1t_2+1\right)t_3^2+\left(-2t_1^2t_2-2t_1t_2^2-8t_1t_2-2t_1-2t_2\right)t_3+\left(t_1^2-2t_1t_2+t_2^2\right)\\ &= (t_1t_2-1)^2t_3^2-2\left(t_1t_2^2+t_1^2t_2+t_1+t_2+4t_1t_2\right)t_3+(t_1-t_2)^2,\end{aligned}$$

which is now a quadratic equation in t_3. We then complete the square to obtain

$$
\begin{aligned}
& (t_1t_2-1)^2\left[t_3^2-2\frac{\left(t_1t_2^2+t_1^2t_2+t_1+t_2+4t_1t_2\right)}{(t_1t_2-1)^2}t_3+\left(\frac{t_1t_2^2+t_1^2t_2+t_1+t_2+4t_1t_2}{(t_1t_2-1)^2}\right)^2\right] \\
= & \left(\frac{(t_1t_2-1)^2t_3-\left(t_1^2t_2+t_1t_2^2+4t_1t_2+t_1+t_2\right)}{(t_1t_2-1)^2}\right)^2=\left(t_3-\frac{t_1^2t_2+t_1t_2^2+4t_1t_2+t_1+t_2}{(t_1t_2-1)^2}\right)^2 \\
= & \frac{\left(t_1t_2^2+t_1^2t_2+t_1+t_2+4t_1t_2\right)^2}{(t_1t_2-1)^4}-(t_1-t_2)^2 \\
= & \frac{\left(t_1t_2^2+t_1^2t_2+t_1+t_2+4t_1t_2\right)^2-(t_1t_2-1)^4(t_1-t_2)^2}{(t_1t_2-1)^4} \\
= & \frac{t_1t_2\left(4t_2-t_1t_2^3+3t_2^2+t_1^2t_2^2-t_1t_2+2\right)\left(4t_1-t_1^3t_2+3t_1^2+t_1^2t_2^2-t_1t_2+2\right)}{(t_1t_2-1)^4}.
\end{aligned}
$$

We multiply both sides of the equation by $(t_1t_2-1)^4$ to obtain a quadratic equation in t_3 in normal form, namely

$$
\begin{aligned}
& \left((t_1t_2-1)^2t_3-t_1^2t_2+t_1t_2^2+4t_1t_2+t_1+t_2\right)^2 \\
= & \; t_1t_2\left(t_1\left(3t_1-t_1^2t_2+4\right)+(t_1t_2-1)^2\right)\left(t_2\left(3t_2-t_1t_2^2+4\right)+(t_1t_2-1)^2\right).
\end{aligned}
$$

Exercise 7.11 (p.93)

Show that the Triple twist formula can be rewritten as

$$(t_1+t_2-t_3-t_1t_2t_3)^2=4t_1t_2(1+t_3)^2.$$

Solution

From the result of Exercise 7.10, the Triple twist formula (p.93) can be expanded and rewritten as

$$t_1^2+t_2^2+t_3^2+t_1^2t_2^2t_3^2-2t_1^2t_2t_3-2t_1t_2^2t_3-2t_1t_2t_3^2-2t_1t_2-2t_1t_3-2t_2t_3-8t_1t_2t_3=0.$$

Add both sides by $4t_1t_2(1+t_3)^2$ to obtain

$$t_1^2+t_2^2+t_3^2+t_1^2t_2^2t_3^2-2t_1^2t_2t_3-2t_1t_2^2t_3+2t_1t_2t_3^2+2t_1t_2-2t_1t_3-2t_2t_3=4t_1t_2(1+t_3)^2$$

and factorise the left hand side to obtain

$$(t_1+t_2-t_3-t_1t_2t_3)^2=4t_1t_2(1+t_3)^2$$

as required.

Exercise 7.12 (p.93)

Show that if three numbers r_1, r_2 and r_3 satisfy

$$r_1+r_2+r_3=r_1r_2r_3$$

and $t_1 \equiv r_1^2$, $t_2 \equiv r_2^2$ and $t_3 \equiv r_3^2$, then $\{t_1, t_2, t_3\}$ satisfy the Triple twist formula.

Solution

If three numbers r_1, r_2 and r_3 satisfy

$$r_1 + r_2 + r_3 = r_1 r_2 r_3,$$

we square both sides and rearrange this equation to get

$$r_1^2 + r_2^2 + r_3^2 - r_1^2 r_2^2 r_3^2 = -2\left(r_1 r_2 + r_1 r_3 + r_2 r_3\right).$$

Define $t_1 \equiv r_1^2$, $t_2 \equiv r_2^2$ and $t_3 \equiv r_3^2$ so that the above equation is rewritten as

$$t_1 + t_2 + t_3 - t_1 t_2 t_3 = -2\left(r_1 r_2 + r_1 r_3 + r_2 r_3\right)$$

and square both sides of this equation to get

$$\begin{aligned}
\left(t_1 + t_2 + t_3 - t_1 t_2 t_3\right)^2 &= 4\left(r_1 r_2 + r_1 r_3 + r_2 r_3\right)^2 \\
&= 4\left(t_1 t_2 + t_1 t_3 + t_2 t_3\right) + 8 r_1 r_2 r_3 \left(r_1 + r_2 + r_3\right) \\
&= 4\left(t_1 t_2 + t_1 t_3 + t_2 t_3 + 2 t_1 t_2 t_3\right),
\end{aligned}$$

which is precisely the Triple twist formula (p.93).