Divine Proportions Solutions
Chapter 7: Triple spread formula
Exercises 7.1 to 7.6

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Exercise 7.1 (p.90)

Show that

$$\begin{aligned} S(a,b,c) &= A(a,b,c) - 4abc \\ &= 2(ab+ac+bc) - \left(a^2+b^2+c^2\right) - 4abc \\ &= 4(ab+ac+bc) - (a+b+c)^2 - 4abc \\ &= 4(1-a)(1-b)(1-c) - (a+b+c-2)^2 \\ &= -\begin{vmatrix} 0 & a & b & 1 \\ a & 0 & c & 1 \\ b & c & 0 & 1 \\ 1 & 1 & 1 & 2 \end{vmatrix}. \end{aligned}$$

Solution

From Exercise 5.7 (p.64), we have

$$\begin{aligned} A(a,b,c) &= (a+b+c)^2 - 2\left(a^2+b^2+c^2\right) \\ &= 2(ab+ac+bc) - \left(a^2+b^2+c^2\right) \\ &= 4(ab+ac+bc) - (a+b+c)^2 \end{aligned}$$

and thus the first three identities all hold. So it remains to show that the last two identities hold.

As

$$\begin{aligned} & 4(1-a)(1-b)(1-c) - (a+b+c-2)^2 \\ =\ & -a^2 - b^2 - c^2 + 2ab + 2ac + 2bc - 4abc \\ =\ & A(a,b,c) - 4abc = S(a,b,c) \end{aligned}$$

and

$$-\begin{vmatrix} 0 & a & b & 1 \\ a & 0 & c & 1 \\ b & c & 0 & 1 \\ 1 & 1 & 1 & 2 \end{vmatrix} = -a^2 - b^2 - c^2 + 2ab + 2ac + 2bc - 4abc = S(a,b,c),$$

the last two identities hold.

Exercise 7.2 (p.90)

Demonstrate the following identity, of importance for the Triangle spread rules (p.219):

$$S(a,b,c) = ((a+b-c)c + (c-a+b)(c-b+a))(1-c) - c(c-(a+b))(1-(a+b)).$$

Solution

We have

$$\begin{aligned} & ((a+b-c)c + (c-a+b)(c-b+a))(1-c) - c(c-(a+b))(1-(a+b)) \\ = \; & -a^2 - b^2 - c^2 + 2ab + 2ac + 2bc - 4abc = S(a,b,c) \end{aligned}$$

as required.

Exercise 7.3 (p.90)

Show that if $\{a,b,0\}$ is a spread triple then $a=b$. Show that if $\{a,b,1\}$ is a spread triple then $a+b=1$.

Solution

If $\{a,b,0\}$ is a spread triple then

$$S(a,b,0) = -(a-b)^2 = 0,$$

for which this is true only when $a=b$ over a general field. On the other hand, if $\{a,b,1\}$ is a spread triple then

$$S(a,b,1) = -(a+b-1)^2 = 0,$$

for which this is true only when $a+b=1$ over a general field.

Exercise 7.4 (p.90)

Show that in the rational or decimal number fields

$$0 \le S(a,a,a)$$

precisely when $a \le \frac{3}{4}$.

Solution

We start with the identity

$$S(a,a,a) = a^2(3-4a).$$

Over the rational or decimal number field, if $a \le \frac{3}{4}$ then it is immediate that $S(a,a,a) \ge 0$ since $a^2 \ge 0$ regardless of the value of a; it remains to show that the converse is true. If $S(a,a,a) \ge 0$, then $a^2(3-4a) \ge 0$ but as $a^2 \ge 0$ for all a in these fields then we must deduce that $3-4a \ge 0$, from which the required result follows.

Exercise 7.5 (p.91)

Show that for any numbers u and v with $u^2 \neq -1$ and $v^2 \neq -1$,

$$\left\{ \frac{u^2}{1+u^2}, \frac{v^2}{1+v^2}, \frac{(u-v)^2}{(1+u^2)(1+v^2)} \right\}$$

is a spread triple. Give an example where not every spread triple is of this form.

Solution

We have that

$$\begin{aligned}
& S\left(\frac{u^2}{1+u^2}, \frac{v^2}{1+v^2}, \frac{(u-v)^2}{(1+u^2)(1+v^2)} \right) \\
= & \left(\frac{u^2}{1+u^2} + \frac{v^2}{1+v^2} + \frac{(u-v)^2}{(1+u^2)(1+v^2)} \right)^2 - 2\left(\left(\frac{u^2}{1+u^2} \right)^2 + \left(\frac{v^2}{1+v^2} \right)^2 + \left(\frac{(u-v)^2}{(1+u^2)(1+v^2)} \right)^2 \right) \\
& -4\left(\frac{u^2}{1+u^2} \right)\left(\frac{v^2}{1+v^2} \right)\left(\frac{(u-v)^2}{(1+u^2)(1+v^2)} \right) \\
= & \frac{4\left[\left(u^2v^2+u^2+v^2-uv\right)^2 - \left(u^4v^4+u^4v^2+u^4-2u^3v+u^2v^4+3u^2v^2-2uv^3+v^4\right) - u^2v^2(u-v)^2 \right]}{(1+u^2)^2(1+v^2)^2} \\
= & \frac{4\left[\left(u^2v^2+u^2+v^2-uv\right)^2 - \left(u^2v^2+u^2+v^2-uv\right)^2 \right]}{(1+u^2)^2(1+v^2)^2} \\
= & 0,
\end{aligned}$$

for which the required result follows.

If, for example, we take the spread triple $\{1,1,0\}$ over the rational number field, where

$$S(1,1,0) = (2)^2 - 2(2) = 0$$

then 0 can only be written in the form $\frac{(u-v)^2}{(1+u^2)(1+v^2)}$, by which $u = v = 0$; however, substituting this into the other two quantities does not give us 1. Thus, we have provided an example of a spread triple that is not of the form above.

Exercise 7.6 (p.91)

Show that as a quadratic equation in s_3, the Triple spread formula is

$$s_3^2 - 2s_3(s_1+s_2-2s_1s_2) + (s_1-s_2)^2 = 0.$$

Show that it can also be written as

$$\begin{aligned}
(s_3-s_1+s_2)^2 &= 4(1-s_1)s_2s_3 \\
(s_3-s_2+s_1)^2 &= 4s_1(1-s_2)s_3 \\
(s_3-s_1-s_2)^2 &= 4s_1s_2(1-s_3)
\end{aligned}$$

and that in normal form it is

$$\left(s_3 - \left(s_1 + s_2 - 2s_1s_2\right)\right)^2 = 4s_1s_2\left(1 - s_1\right)\left(1 - s_2\right).$$

Solution

We may rewrite the Triple spread formula as

$$S\left(s_1, s_2, s_3\right) = \left(2s_1 + 2s_2 - 4s_1s_2\right)s_3 - s_3^2 + \left(2s_1s_2 - s_1^2 - s_2^2\right) = 0,$$

for which we simplify this to get

$$s_3^2 - 2s_3\left(s_1 + s_2 - 2s_1s_2\right) + \left(s_1 - s_2\right)^2 = 0$$

as required.

From here, we may add both sides by $4\left(1 - s_1\right)s_2s_3$ to get

$$\begin{aligned} 4\left(1 - s_1\right)s_2s_3 &= s_3^2 - 2s_3\left(s_1 + s_2 - 2s_1s_2\right) + \left(s_1 - s_2\right)^2 + 4\left(1 - s_1\right)s_2s_3 \\ &= s_3^2 + \left(-2\left(s_1 + s_2 - 2s_1s_2\right) + 4\left(1 - s_1\right)s_2\right)s_3 + \left(s_1 - s_2\right)^2 \\ &= s_3^2 - 2\left(s_1 - s_2\right)s_3 + \left(s_1 - s_2\right)^2 \\ &= \left(s_3 - s_1 + s_2\right)^2 \end{aligned}$$

as required. We add both sides of the same equation by $4s_1\left(1 - s_2\right)s_3$ and $4s_1s_2\left(1 - s_3\right)$ to respectively get

$$\left(s_3 - s_1 + s_2\right)^2 = 4s_1\left(1 - s_2\right)s_3 \quad \text{and} \quad \left(s_3 - s_1 - s_2\right)^2 = 4s_1s_2\left(1 - s_3\right).$$

Finally, from the formulation of the Triple spread formula as a quadratic equation in s_3, we may complete the square to obtain

$$\begin{aligned} 0 &= s_3^2 - 2s_3\left(s_1 + s_2 - 2s_1s_2\right) + \left(s_1 - s_2\right)^2 \\ &= \left(s_3 - \left(s_1 + s_2 - 2s_1s_2\right)\right)^2 + \left(s_1 - s_2\right)^2 - \left(s_1 + s_2 - 2s_1s_2\right)^2, \end{aligned}$$

which then simplifies to

$$\left(s_3 - \left(s_1 + s_2 - 2s_1s_2\right)\right)^2 = \left(s_1 + s_2 - 2s_1s_2\right)^2 - \left(s_1 - s_2\right)^2 = 4s_1s_2\left(1 - s_1\right)\left(1 - s_2\right)$$

as required.