Divine Proportions Solutions

Chapter 6: Spread

Exercises 6.1 to 6.7

Dr. Gennady Arshad Notowidigdo, PhD

Exercise 6.1 (p.78)

Show that if the spread s, cross c and twist t are defined between two lines, then

$$t = \frac{s}{c} = \frac{s}{1-s} = \frac{1-c}{c}, \quad s = \frac{t}{1+t} \quad \text{and} \quad c = \frac{1}{1+t}.$$

Solution

By the definition of the spread, cross and twist between two lines $l_1 = \langle a_1 : b_1 : c_1 \rangle$ and $l_2 = \langle a_2 : b_2 : c_2 \rangle$, we have

$$\frac{s}{c} = \frac{(a_1b_2 - a_2b_1)^2}{(a_1^2+b_1^2)(a_2^2+b_2^2)} \div \frac{(a_1b_1 + a_2b_2)^2}{(a_1^2+b_1^2)(a_2^2+b_2^2)} = \left(\frac{a_1b_2 - a_2b_1}{a_1b_1 + a_2b_2}\right)^2 = t.$$

From there, the Spread plus cross theorem (p.76) implies that

$$t = \frac{s}{1-s} = \frac{1-c}{c},$$

at which point further rearrangement gives us our desired results.

Exercise 6.2 (p.79)

Show that the converse of the Complementary spreads theorem (p.79) does not hold.

Solution

For lines $l_1 \equiv \langle a_1 : b_1 : c_1 \rangle$, $l_2 \equiv \langle a_2 : b_2 : c_2 \rangle$ and $l_3 \equiv \langle a_3 : b_3 : c_3 \rangle$ with l_3 being non-null, set

$$s_1 \equiv s(l_2, l_3) = \frac{(a_2b_3 - a_3b_2)^2}{(a_2^2+b_2^2)(a_3^2+b_3^2)} \quad \text{and} \quad s_2 \equiv s(l_1, l_3) = \frac{(a_1b_3 - a_3b_1)^2}{(a_1^2+b_1^2)(a_3^2+b_3^2)},$$

where we express s_1 and s_2 in terms of the parameters of the lines using the definition of the spread between two lines. If s_1 and s_2 are complementary, then we have that

$$\begin{aligned} s_1 + s_2 &= \frac{(a_2b_3 - a_3b_2)^2}{(a_2^2+b_2^2)(a_3^2+b_3^2)} + \frac{(a_1b_3 - a_3b_1)^2}{(a_1^2+b_1^2)(a_3^2+b_3^2)} \\ &= \frac{(a_1^2+b_1^2)(a_2b_3 - a_3b_2)^2 + (a_2^2+b_2^2)(a_1b_3 - a_3b_1)^2}{(a_1^2+b_1^2)(a_2^2+b_2^2)(a_3^2+b_3^2)} \\ &= 1. \end{aligned}$$

This equality is true precisely when

$$\left(a_1^2+b_1^2\right)\left(a_2b_3-a_3b_2\right)^2+\left(a_2^2+b_2^2\right)\left(a_1b_3-a_3b_1\right)^2=\left(a_1^2+b_1^2\right)\left(a_2^2+b_2^2\right)\left(a_3^2+b_3^2\right),$$

that is when

$$\begin{aligned} 0 &= \left(a_1^2+b_1^2\right)\left(a_2^2+b_2^2\right)\left(a_3^2+b_3^2\right)-\left(a_1^2+b_1^2\right)\left(a_2b_3-a_3b_2\right)^2-\left(a_2^2+b_2^2\right)\left(a_1b_3-a_3b_1\right)^2 \\ &= \left(a_1a_2+b_1b_2\right)\left(a_1a_2a_3^2-a_1a_2b_3^2-a_3^2b_1b_2+b_1b_2b_3^2+2a_1a_3b_2b_3+2a_2a_3b_1b_3\right). \end{aligned}$$

This is true either when $a_1a_2+b_1b_2=0$, in which case by the definition of the cross between two lines we get that l_1 and l_2 are perpendicular, or when the other factor is zero, which does not necessarily imply perpendicularity of l_1 and l_2. Therefore, the converse of the Corresponding spreads theorem (p.79) does not generally hold.

Exercise 6.3 (p.82)

Use the Quadrea spread theorem (p. 82) to show that if a triangle $\overline{A_1A_2A_3}$ has quadrances Q_1, Q_2 and Q_3, spreads s_1, s_2 and s_3, and quadrea $\mathcal{A}$, then

$$\frac{s_1}{Q_1}=\frac{s_2}{Q_2}=\frac{s_3}{Q_3}=\frac{\mathcal{A}}{4Q_1Q_2Q_3}.$$

Solution

The Spread law (p.80) already gives us the first three equalities for any triangle $\overline{A_1A_2A_3}$. By the Quadrea spread theorem (p.82), we have that

$$s_1=\frac{\mathcal{A}}{4Q_2Q_3},\quad s_2=\frac{\mathcal{A}}{4Q_1Q_3}\quad\text{and}\quad s_3=\frac{\mathcal{A}}{4Q_1Q_2}.$$

Divide s_1, s_2 and s_3 by Q_1, Q_2 and Q_3 respectively to obtain our desired result.

Exercise 6.4 (p.82)

Suppose the null triangle $\overline{A_1A_2A_3}$ has quadrances Q_1, Q_2 and Q_3, with $Q_3\equiv 0$. Show that the quadrea of the triangle is

$$\mathcal{A}=-\left(Q_1-Q_2\right)^2$$

and that

$$s_3=-\frac{\left(Q_1-Q_2\right)^2}{4Q_1Q_2}.$$

Solution

If $Q_3=0$, then by the definition of the quadrea

$$\begin{aligned} \mathcal{A} &= \left(Q_1+Q_2+0\right)^2-2\left(Q_1^2+Q_2^2+0^2\right) \\ &= Q_1^2+2Q_1Q_2+Q_2^2-2Q_1^2-2Q_2^2 \\ &= -Q_1^2+2Q_1Q_2-Q_2^2 \\ &= -\left(Q_1-Q_2\right)^2 \end{aligned}$$

as required. So, by the Quadrea spread theorem (p.82), we have that

$$s_3 = \frac{\mathcal{A}}{4Q_1Q_2} = -\frac{(Q_1 - Q_2)^2}{4Q_1Q_2}$$

as required.

Exercise 6.5 (p.82)

Suppose the null triangle $\overline{A_1A_2A_3}$ has quadrances $Q_1 \neq 0$ and $Q_2 = Q_3 \equiv 0$. Show that the quadrea is

$$\mathcal{A} = -Q_1^2.$$

Solution

Now, if $Q_2 = Q_3 = 0$, then by the definition of the quadrea

$$\mathcal{A} = (Q_1 + 0 + 0)^2 - 2\left(Q_1^2 + 0 + 0^2\right) = Q_1^2 - 2Q_1^2 = -Q_1^2$$

as required.

Exercise 6.6 (p.82)

In the rational or decimal number fields, show that if a triangle has quadrances Q_1, Q_2 and Q_3, then

$$(Q_1 + Q_2 - Q_3)(Q_1 + Q_2 - Q_3) \leq 4Q_1Q_2.$$

Solution

As all squares of rational or decimal numbers are non-negative, the Quadrea theorem (p.68) assures us that $\mathcal{A}$ is non-negative. Since

$$\mathcal{A} = 4Q_1Q_2 - (Q_1 + Q_2 - Q_3)^2$$

our required result immediately follows.

Exercise 6.7 (p.84)

Formalise a corresponding result to the Spread from points and Cross from points theorems (p.84) for twists.

Solution

From the Spread from points and Cross from points theorems (p.84), we use the results of Exercise 6.1 (p.78) to obtain

$$t(A_1A_3, A_2A_3) = \left(\frac{(y_1 - y_3)(x_3 - x_2) - (y_2 - y_3)(x_3 - x_1)}{(y_1 - y_3)(y_2 - y_3) + (x_3 - x_1)(x_3 - x_2)}\right)^2$$

for points $A_1 \equiv [x_1, y_1]$, $A_2 \equiv [x_2, y_2]$ and $A_3 \equiv [x_3, y_3]$.