Divine Proportions Solutions
Chapter 5: Quadrance
Exercises 5.7 to 5.12

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Exercise 5.7 (p.64)

Show that

$$\begin{aligned} A(a,b,c) &= 4ab-(a+b-c)^2 \\ &= 2(ab+bc+ca)-\left(a^2+b^2+c^2\right) \\ &= 4(ab+bc+ca)-(a+b+c)^2 \\ &= \begin{vmatrix} 2a & a+b-c \\ a+b-c & 2b \end{vmatrix} \\ &= -\begin{vmatrix} 0 & a & b & 1 \\ a & 0 & c & 1 \\ b & c & 0 & 1 \\ 1 & 1 & 1 & 0 \end{vmatrix}. \end{aligned}$$

Solution

Given that

$$A(a,b,c) \equiv (a+b+c)^2 - 2\left(a^2+b^2+c^2\right)$$

the first statement here has already been proven in the proof of the Triple quad formula (p.63). The fourth statement immediately follows from the calculation of the determinant, once the first statement is established. By calculating the given determinant, the fifth statement becomes

$$\begin{aligned} -\begin{vmatrix} 0 & a & b & 1 \\ a & 0 & c & 1 \\ b & c & 0 & 1 \\ 1 & 1 & 1 & 0 \end{vmatrix} &= -a^2+2ab+2ac-b^2+2bc-c^2 \\ &= \left(a^2+b^2+c^2\right)+2ab+2ac+2bc-2\left(a^2+b^2+c^2\right) \\ &= (a+b+c)^2-2\left(a^2+b^2+c^2\right) \\ &= A(a,b,c) \end{aligned}$$

so that the second statement immediately follows from this as well. It remains now to prove that the third statement follows. From the second statement, we have

$$\begin{aligned} 2\left(ab+bc+ca\right)-\left(a^2+b^2+c^2\right) &= 2\left(ab+bc+ca\right)-\left(a^2+b^2+c^2+2\left(ab+bc+ca\right)\right)+2\left(ab+bc+ca\right) \\ &= 4\left(ab+bc+ca\right)-\left(a+b+c\right)^2 \end{aligned}$$

and thus we have all our required results.

Exercise 5.8 (p.64)

Show that if three quantities a, b and c satisfy one of the relations $a \pm b = \pm c$, then $A \equiv a^2$, $B \equiv b^2$ and $C \equiv c^2$ form a quad triple $\{A, B, C\}$.

Solution

Since the squares of c and $-c$ are equal, it is sufficient to prove that

$$\left\{a^2, b^2, (a+b)^2\right\} \quad \text{and} \quad \left\{a^2, b^2, (a-b)^2\right\}$$

are both quad triples. Using the first statement from Exercise 5.7 (p.64), we have

$$\begin{aligned} A\left(a^2, b^2, (a+b)^2\right) &= 4a^2b^2-\left(a^2+b^2-(a+b)^2\right)^2 \\ &= (2ab)^2-(-2ab)^2 = 0, \end{aligned}$$

and similarly

$$\begin{aligned} A\left(a^2, b^2, (a-b)^2\right) &= 4a^2b^2-\left(a^2+b^2-(a-b)^2\right)^2 \\ &= (2ab)^2-(2ab)^2 = 0. \end{aligned}$$

The required result immediately follows.

Exercise 5.9 (p.64)

Show that in general not every quad triple is of the form

$$\left\{a^2, b^2, (a+b)^2\right\}$$

for some numbers a and b.

Solution

This result follows directly from Exercise 5.8 (p. 64), as we have shown that $\left\{a^2, b^2, (a-b)^2\right\}$ is a quad triple.

Exercise 5.10 (p.64)

Show that as a quadratic equation in Q_3, the Triple quad formula is

$$Q_3^2 - 2\left(Q_1 + Q_2\right) Q_3 + \left(Q_1 - Q_3\right)^2 = 0.$$

Solution

The Triple quad formula can be re-written as

$$\begin{aligned} 0 &= \left(Q_1 + Q_2 + Q_3\right)^2 - 2\left(Q_1^2 + Q_2^2 + Q_3^2\right) \\ &= 2Q_1Q_2 + 2Q_1Q_3 + 2Q_2Q_3 - Q_1^2 - Q_2^2 - Q_3^2. \end{aligned}$$

Multiply both sides of this equation by -1 and rearrange to obtain

$$\begin{aligned} 0 &= Q_1^2 + Q_2^2 + Q_3^2 - 2Q_1Q_2 - 2Q_1Q_3 - 2Q_2Q_3 \\ &= Q_3^2 - 2\left(Q_1 + Q_2\right) Q_3 + \left(Q_1 - Q_3\right)^2, \end{aligned}$$

as required.

Exercise 5.11 (p.64)

Show that the two quadratic equations in x

$$\begin{aligned} \left(x - p_1\right)^2 &= q_1 \\ \left(x - p_2\right)^2 &= q_2 \end{aligned}$$

are compatible precisely when $\left\{q_1, q_2, \left(p_1 - p_2\right)^2\right\}$ is a quad triple.

Solution

By the Quadratic compatability theorem (p.33), the quadratic equations

$$\left(x - p_1\right)^2 = q_1 \quad \text{and} \quad \left(x - p_2\right)^2 = q_2$$

above in x are compatible precisely when

$$\left(\left(p_1 - p_2\right)^2 - \left(q_1 + q_2\right)\right)^2 = 4q_1q_2,$$

So, we have

$$4q_1q_2 - \left(q_1 + q_2 - \left(p_1 - p_2\right)^2\right)^2 = A\left(q_1, q_2, \left(p_1 - p_2\right)^2\right) = 0$$

from the first statement of Exercise 5.7 (p.64). The required result immediately follows.

Exercise 5.12 (p.65)

Show that if $\overline{A_1A_2A_3}$ is a right triangle with right vertex at A_3, then it is impossible for either Q_1 or Q_2 to be zero. Give an example to show that in some fields it is possible for Q_3 to be zero.

Solution

For a right triangle $\overline{A_1A_2A_3}$ with right vertex at A_3, we are supposing that

$$Q_1 \equiv Q(A_2, A_3), \quad Q_2 \equiv Q(A_1, A_3) \quad \text{and} \quad Q_3 \equiv Q(A_1, A_2),$$

so that Pythagoras' theorem (p. 65) gives us the identity $Q_1 + Q_2 = Q_3$. If, say, $Q_1 \equiv 0$, then we must have that $Q_2 = Q_3$ in which case the only possibility of this occuring is if:

1. A_2 and A_3 are not distinct points; or
2. the points A_2 and A_3 lie on a circle of squared radius Q_2 or Q_3, centred at A_1.

In both cases, we have a contradiction since either case makes the vertex at A_3 no longer a right vertex; thus, it is impossible for Q_1 to be 0. Parallel reasoning also leads us to deduce that Q_2 cannot be 0.

For a right triangle over $\mathbb{F}_{13}$ with vertices $[0,0]$, $[2,0]$ and $[0,3]$, it is easy to see that $2^2 + 3^2 = 4 + 9 = 13 \equiv 0$; thus, it is possible for Q_3 to be zero in some fields.