Extending vector products to general inner product spaces

G A Notowidigdo
School of Mathematics and Statistics
UNSW Sydney
Sydney, NSW, Australia
g.notowidigdo@unsw.edu.au
gnotowidigdo@gmail.com

Abstract

This paper extends the Euclidean vector product to a general inner product space with the inner product defined by a general non-degenerate symmetric bilinear form which will be represented by a 3×3 invertible symmetric matrix. From there, the usual definitions of scalar and vector triple and quadruple products are extended to this general inner product space and known results of Binet, Cauchy, Jacobi and Lagrange are generalised.

Keywords: scalar product; vector product; symmetric bilinear form; inner product space
2010 MSC Numbers: 15A63, 11E39, 11E88

1 Introduction

The **Euclidean vector product** of two vectors $v = (v_1, v_2, v_3)$ and $w = (w_1, w_2, w_3)$ in the three-dimensional vector space $\mathbb{R}^3$ over the set of real numbers $\mathbb{R}$ is defined as the vector

$$\langle v, w \rangle = v \times w = (v_2 w_3 - v_3 w_2, v_3 w_1 - v_1 w_3, v_1 w_2 - v_2 w_1).$$

This definition was introduced in component form by Lagrange in [19] to study the tetrahedron in three dimensions, and then formally presented by Gibbs in [11] as well as Heaviside [14, pp. 132-305] to simplify the works of Clifford [6] and Hamilton [12] on quaternion products.

In this paper, we aim to extend this definition to a general inner product space $\mathbb{F}^3$ over a general field $\mathbb{F}$ not of characteristic 2, where the inner product on this space is given by a non-degenerate symmetric bilinear form on $\mathbb{F}^3$ which can be represented by a 3×3 invertible symmetric matrix B. We will call such an inner product the B**-scalar product**, and we denote them in this paper for vectors u and v in $\mathbb{F}^3$ by $[u, v]_B$ Given the above definition of the Euclidean vector product, now over $\mathbb{F}$ rather than $\mathbb{R}$, we can then define the B**-vector product** as

$$\langle v, w \rangle_B = \langle v, w \rangle \operatorname{adj} B$$

where adj is the operator denoting the **adjugate matrix**.

From this, we can the define for vectors u, v, w and x in $\mathbb{F}^3$:

- the B-**scalar triple product**

$$[u, v, w]_B = [u, \langle v, w\rangle_B]_B$$

- the B-**vector triple product**

$$\langle u, v, w\rangle_B = \langle u, \langle v, w\rangle_B\rangle_B$$

- the B-**scalar quadruple product**

$$[u, v; w, x]_B = [\langle u, v\rangle_B, \langle w, x\rangle_B]_B$$

- and the B-**vector quadruple product**

$$\langle u, v; w, x\rangle_B = \langle\langle u, v\rangle_B, \langle w, x\rangle_B\rangle_B.$$

These four definitions extend the usual definitions of the scalar and vector triple and quadruple products from [11] to the general inner product space prescribed above. From this, we can generalise known results of Binet [2], Cauchy [4], Jacobi [16] and Lagrange [19] to this general inner product space and we can also rely on [21] for additional interesting results related to these triple and quadruple products.

We will conclude this paper by briefly discussing the scope of applications of the extension of vector products to general inner product spaces.

2 Symmetric bilinear forms and general inner product spaces

Consider the three-dimensional vector space $\mathbb{F}^3$ over a general field $\mathbb{F}$ not of characteristic 2. The objects of $\mathbb{F}^3$ are **vectors**, which we write in row form, like (x, y, z).

A 3×3 symmetric matrix

$$B = \begin{pmatrix} a_1 & b_3 & b_2 \\ b_3 & a_2 & b_1 \\ b_2 & b_1 & a_3 \end{pmatrix} \tag{1}$$

determines a **symmetric bilinear form** on $\mathbb{F}^3$ defined and denoted by

$$[v, w]_B = vBw^T$$

and under such a symmetric bilinear form $\mathbb{F}^3$ then becomes an **inner product space**. We say then that the matrix B represents the symmetric bilinear form on $\mathbb{F}^3$ and thus by extension the inner product space itself. We may call this kind of operation the B-**scalar product**, where in the case that B is the 3×3 identity matrix, it corresponds to the usual **Euclidean scalar product** on two vectors [11, pp. 55-57].

The symmetric bilinear form on $\mathbb{F}^3$ is **non-degenerate** if and only if for all vectors v in $\mathbb{F}^3$ we have that $[v, w]_B = 0$ implies $w = \mathbf{0}$, where $\mathbf{0}$ is the zero vector in $\mathbb{F}^3$; this is equivalent to the condition that the matrix B representing the symmetric bilinear form is invertible. We will assume that the symmetric bilinear form is non-degenerate throught this paper.

We can then define the B-**quadratic form** on the inner product space $\mathbb{F}^3$ equipped with the B-scalar product by

$$Q_B(v) = [v, v]_B$$

for all vectors v, and we say that a vector v is a B-**null vector** if $Q_B(v) = 0$. We then have for all vectors v and w in $\mathbb{F}^3$ and a scalar λ in $\mathbb{F}$ that

$$Q_B(\lambda v) = \lambda^2 Q_B(v)$$

which consequently means that

$$Q_B(-v) = Q_B(v).$$

We also have that

$$Q_B(v + w) = Q_B(v) + Q_B(w) + 2[v, w]_B$$

and

$$Q_B(v - w) = Q_B(v) + Q_B(w) - 2[v, w]_B$$

so that we may express the symmetric bilinear form in terms of its associated quadratic form through the **polarisation formulas** [13]

$$[v, w]_B = \frac{Q_B(v + w) - Q_B(v) - Q_B(w)}{2} = \frac{Q_B(v) + Q_B(w) - Q_B(v - w)}{2}.$$

Two vectors v and w in $\mathbb{F}^3$ are **orthogonal** with respect to the B-scalar product if $[v, w]_B = 0$ and we denote this relationship by $v \perp_B w$; by the above polarisation formulas this is equivalent to the condition that

$$Q_B(v + w) = Q_B(v) + Q_B(w) = Q_B(v - w).$$

3 Vector product on inner product space

Given two vectors $v_1 = (x_1, y_1, z_1)$ and $v_2 = (x_2, y_2, z_2)$ in an inner product space $\mathbb{F}^3$ equipped with the usual Euclidean scalar product, the **Euclidean vector product** [11, pp. 60-62] of v_1 and v_2 is

$$\langle v_1, v_2 \rangle = v_1 \times v_2 = (y_1 z_2 - y_2 z_1, x_2 z_1 - x_1 z_2, x_1 y_2 - x_2 y_1).$$

We now extend this notion to the case of a general inner product space.

Let v_1, v_2 and v_3 be vectors in $\mathbb{V}^3$, and let $M = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$ be the matrix with these vectors as rows.

We define the **adjugate** of M (see [10, p. 82] and [25, p. 232]) to be the matrix

$$\operatorname{adj} M = \begin{pmatrix} \langle v_2, v_3 \rangle \\ \langle v_3, v_1 \rangle \\ \langle v_1, v_2 \rangle \end{pmatrix}^T.$$

If the 3×3 matrix M is invertible, then the adjugate is characterized by the equation

$$\frac{1}{(\det M)} \operatorname{adj} M = M^{-1}.$$

One can then show in this case that the two properties

$$\operatorname{adj}(MN) = (\operatorname{adj} N)(\operatorname{adj} M)$$

and

$$M(\operatorname{adj} M) = (\operatorname{adj} M) M = (\det M) I_3$$

hold, where I_3 is the 3×3 identity matrix, and in fact they hold more generally for arbitrary 3×3 matrices M and N. In the invertible case we have also that

$$\begin{aligned} \operatorname{adj}(\operatorname{adj} M) &= \det(\operatorname{adj} M)(\operatorname{adj} M)^{-1} \\ &= \det\left((\det M) M^{-1}\right)\left((\det M) M^{-1}\right)^{-1} \\ &= (\det M)^3 \left(\det M^{-1}\right)(\det M)^{-1} M \\ &= (\det M) M. \end{aligned}$$

For the fixed symmetric matrix B from (1), we write and denote

$$\operatorname{adj} B = \begin{pmatrix} a_2 a_3 - b_1^2 & b_1 b_2 - a_3 b_3 & b_1 b_3 - a_2 b_2 \\ b_1 b_2 - a_3 b_3 & a_1 a_3 - b_2^2 & b_2 b_3 - a_1 b_1 \\ b_1 b_3 - a_2 b_2 & b_2 b_3 - a_1 b_1 & a_1 a_2 - b_3^2 \end{pmatrix} = \begin{pmatrix} \alpha_1 & \beta_3 & \beta_2 \\ \beta_3 & \alpha_2 & \beta_1 \\ \beta_2 & \beta_1 & \alpha_3 \end{pmatrix}. \tag{2}$$

For the inner product space $\mathbb{F}^3$ equipped with the B-scalar product, define the B-**vector product** of vectors v_1 and v_2 to be the vector

$$\langle v_1, v_2 \rangle_B = \langle v_1, v_2 \rangle \operatorname{adj} B$$

where $\langle v_1, v_2 \rangle$ is the Euclidean vector product of v_1 and v_2 from above. The motivation for this definition is given by the following theorem, based on a similar result given in [7].

Theorem 1 (Adjugate vector product theorem) *Let v_1, v_2 and v_3 be vectors in $\mathbb{V}^3$, and let $M = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$ be the matrix with these vectors as rows. Then for any 3×3 invertible symmetric matrix*

B, we have that

$$\operatorname{adj}(MB) = \begin{pmatrix} \langle v_2, v_3 \rangle_B \\ \langle v_3, v_1 \rangle_B \\ \langle v_1, v_2 \rangle_B \end{pmatrix}^T .$$

Proof. By the definition of adjugate matrix, $\operatorname{adj} M$ is

$$\operatorname{adj} M = \begin{pmatrix} \langle v_2, v_3 \rangle \\ \langle v_3, v_1 \rangle \\ \langle v_1, v_2 \rangle \end{pmatrix}^T .$$

Since $\operatorname{adj}(MB) = \operatorname{adj} B \operatorname{adj} M$ and B is symmetric, we get

$$\begin{aligned} (\operatorname{adj}(MB))^T &= (\operatorname{adj} M)^T \operatorname{adj} B = \begin{pmatrix} \langle v_2, v_3 \rangle \\ \langle v_3, v_1 \rangle \\ \langle v_1, v_2 \rangle \end{pmatrix} \operatorname{adj} B \\ &= \begin{pmatrix} \langle v_2, v_3 \rangle \operatorname{adj} B \\ \langle v_3, v_1 \rangle \operatorname{adj} B \\ \langle v_1, v_2 \rangle \operatorname{adj} B \end{pmatrix} = \begin{pmatrix} \langle v_2, v_3 \rangle_B \\ \langle v_3, v_1 \rangle_B \\ \langle v_1, v_2 \rangle_B \end{pmatrix} . \end{aligned}$$

Now take the matrix transpose on both sides to get the required result. ■

The usual linearity and anti-symmetric properties of the Euclidean vector product (shown in [1, pp. 142-143]) will hold for B-vector products.

3.1 Scalar triple products over general inner product spaces

Given the inner product space $\mathbb{F}^3$ equipped with the usual Euclidean scalar product, the **Euclidean scalar triple product** of three vectors v_1, v_2 and v_3 in $\mathbb{F}^3$ is defined in [11, pp. 68-71] as

$$[v_1, v_2, v_3] = [v_1, \langle v_2, v_3 \rangle] = \det \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} .$$

We can generalise this definition for a general inner product space $\mathbb{F}^3$ equipped with the B-scalar product. So, we now define the **B-scalar triple product** of three vectors v_1, v_2 and v_3 in $\mathbb{F}^3$ to be

$$[v_1, v_2, v_3]_B = [v_1, \langle v_2, v_3 \rangle_B]_B .$$

The following result allows for the evaluation of the B-scalar triple product in terms of determinants, which generalises well-known formulas relating Euclidean scalar products and determinants, such as in [21, p. 104].

Theorem 2 (Generalised scalar triple product theorem) *Let* $M = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$ *for vectors* v_1, v_2

and v_3 in an inner product space $\mathbb{F}^3$ equipped with the B-scalar product. Then, we have that

$$[v_1, v_2, v_3]_B = (\det B)\,(\det M)\,.$$

Proof. From the definitions of the B-scalar product, B-vector product and the B-scalar triple product, we have

$$\begin{aligned}[v_1, v_2, v_3]_B &= v_1 B\left(\langle v_2, v_3\rangle \operatorname{adj} B\right)^T \\ &= v_1\left(B \operatorname{adj} B\right)\langle v_2, v_3\rangle^T\,.\end{aligned}$$

As $\operatorname{adj} B = (\det B)\,B^{-1}$ and $[v_1, v_2, v_3] = \det M$ by the definition of the Euclidean scalar triple product, we get

$$\begin{aligned}[v_1, v_2, v_3]_B &= (\det B)\,v_1\,\langle v_2, v_3\rangle^T \\ &= (\det B)\,[v_1, v_2, v_3] \\ &= (\det B)\,(\det M)\end{aligned}$$

as required. ∎

We can now relate B-vector products to orthogonality with respect to the B-scalar product.

Corollary 3 *The vectors v and w in $\mathbb{V}^3$ are both orthogonal to $\langle v, w\rangle_B$ with respect to the B-scalar product, i.e.*

$$v \perp_B (v \times_B w) \quad \text{and} \quad w \perp_B (v \times_B w)\,.$$

Proof. By the Generalised scalar triple product theorem,

$$[v, \langle v, w\rangle_B]_B = [v, v, w]_B = (\det B)\det\begin{pmatrix} v \\ v \\ w \end{pmatrix} = 0.$$

Similarly, $[w, v, w]_B = 0$ and so we have the desired result by definition of orthogonality with respect to the B-scalar product. ∎

The above corollary generalise the well-known fact that the Euclidean vector product gives a vector perpendicular to the two operands in the Euclidean sense to a general inner product space.

We could also rearrange the ordering of B-scalar triple products as follows.

Corollary 4 *For vectors v_1, v_2 and v_3 in $\mathbb{V}^3$,*

$$\begin{gathered}[v_1, v_2, v_3]_B = [v_2, v_3, v_1]_B = [v_3, v_1, v_2]_B \\ = -\,[v_1, v_3, v_2]_B = -\,[v_2, v_1, v_3]_B = -\,[v_3, v_2, v_1]_B\,.\end{gathered}$$

Proof. This follows from the corresponding relations for $[v_1, v_2, v_3]$, or equivalently the transformation properties of the determinant upon permutation of rows. ∎

3.2 Vector triple products over general inner product spaces

Recall that the **Euclidean vector triple product** of three vectors v_1, v_2 and v_3 in the inner product space $\mathbb{F}^3$ equipped with the usual Euclidean scalar product (from [11, pp. 71-75]) is

$$\langle v_1, v_2, v_3 \rangle = \langle v_1, \langle v_2, v_3 \rangle \rangle .$$

The B-**vector triple product** of the same vectors on a more general inner product space equipped with the B-scalar product is similarly defined to be

$$\langle v_1, v_2, v_3 \rangle_B = \langle v_1, \langle v_2, v_3 \rangle_B \rangle_B .$$

We can evaluate this by generalising a classical result of Lagrange [19] from the Euclidean vector triple product to B-vector triple products, following the general lines of argument of [5] and [24, pp. 28-29]; the proof is surprisingly complicated.

Theorem 5 (Generalised Lagrange formula) *For vectors v_1, v_2 and v_3 in the inner product space $\mathbb{F}^3$ equipped with the B-scalar product,*

$$\langle v_1, v_2, v_3 \rangle_B = (\det B) \left[[v_1, v_3]_B \, v_2 - [v_1, v_2]_B \, v_3 \right] .$$

Proof. Let $w = \langle v_1, v_2, v_3 \rangle_B$. If v_2 and v_3 are linearly dependent, then $\langle v_2, v_3 \rangle_B = \mathbf{0}$ and thus $\langle v_1, v_2, v_3 \rangle_B = \mathbf{0}$. Furthermore, we are able to write one of them as a scalar multiple of the other, which implies that

$$[v_1, v_3]_B \, v_2 - [v_1, v_2]_B \, v_3 = \mathbf{0}$$

and thus the required result holds. So we may suppose that v_2 and v_3 are linearly independent. From Corollary 3, $\langle v_2, v_3 \rangle_B \perp_B w$ and thus

$$v_2 \perp_B \langle v_2, v_3 \rangle_B \quad \text{and} \quad v_3 \perp_B \langle v_2, v_3 \rangle_B .$$

As w is parallel to v_2 and v_3, we can deduce that w is equal to some linear combination of v_2 and v_3. So, for some scalars α and β in $\mathbb{F}$, we have

$$w = \alpha v_2 + \beta v_3.$$

Furthermore, since $v_1 \perp_B w$, the definition of orthogonality with respect to the B-scalar product implies that

$$[w, v_1]_B = \alpha \, [v_1, v_2]_B + \beta \, [v_1, v_3]_B = 0.$$

This equality is true precisely when $\alpha = \lambda \, [v_1, v_3]_B$ and $\beta = -\lambda \, [v_1, v_2]_B$, for some non-zero scalar λ in $\mathbb{F}$. Hence,

$$w = \lambda \left[[v_1, v_3]_B \, v_2 - [v_1, v_2]_B \, v_3 \right] .$$

To proceed, we first want to prove that λ is independent of the choices v_1, v_2 and v_3, so that we can compute w for arbitrary v_1, v_2 and v_3. First, suppose that λ is dependent on v_1, v_2 and v_3, so that we may define $\lambda = \lambda(v_1, v_2, v_3)$. Given another vector d in $\mathbb{F}^3$, we have

$$[w, d]_B = \lambda(v_1, v_2, v_3)\left[[v_1, v_3]_B [v_2, d]_B - [v_1, v_2]_B [v_3, d]_B\right]. \tag{3}$$

Directly substituting the definition of w, we use the Generalised scalar triple product theorem to obtain

$$[w, d]_B = \langle v_1, \langle v_2, v_3\rangle_B\rangle_B \cdot_B d = \left[v_1, \langle\langle v_2, v_3\rangle_B, d\rangle_B\right]_B = -\left[v_1, \langle d, v_2, v_3\rangle_B\right]_B.$$

Based on our calculations of w, we then deduce that

$$\begin{aligned} -\left[v_1, \langle d, v_2, v_3\rangle_B\right]_B &= -\left[v_1, \lambda(d, v_2, v_3)\left[[d, v_3]_B v_2 - [d, v_2]_B v_3\right]\right]_B \\ &= \lambda(d, v_2, v_3)\left[[v_1, v_3]_B [v_2, d]_B - [v_1, v_2]_B [v_3, d]_B\right]. \end{aligned} \tag{4}$$

Comparing (3) and (4), we deduce that $\lambda(v_1, v_2, v_3) = \lambda(d, v_2, v_3)$ and hence λ must be independent of the choice of v_1. With this, now suppose instead that $\lambda = \lambda(v_2, v_3)$, so that

$$[w, d]_B = \lambda(v_2, v_3)\left[[v_1, v_3]_B [v_2, d]_B - [v_1, v_2]_B [v_3, d]_B\right] \tag{5}$$

for a vector d in $\mathbb{F}^3$. By direct substitution of w, we use the Generalised scalar triple product theorem to obtain

$$[w, d]_B = \left[\langle v_1, \langle v_2, v_3\rangle_B\rangle_B, d\right]_B = \left[[v_2, v_3]_B, [d, v_1]_B\right]_B = \left[v_2, \langle v_3, d, v_1\rangle_B\right]_B.$$

Similarly, based on the calculations of w previously, we have

$$\begin{aligned} \left[v_2, \langle v_3, d, v_1\rangle_B\right]_B &= \left[v_2, \lambda(v_2, v_3)\left([v_1, v_3]_B d - [v_3, d]_B v_1\right)\right]_B \\ &= \lambda(d, v_1)\left[[v_1, v_3]_B [v_2, d]_B - [v_1, v_2]_B [v_3, d]_B\right]. \end{aligned} \tag{6}$$

Comparing (5) and (6), we deduce that $\lambda(v_2, v_3) = \lambda(d, v_1)$ and conclude that λ is indeed independent of v_2 and v_3, in addition to v_1. So, we substitute any choice of vectors v_1, v_2 and v_3 in order to find λ. With this, suppose that $v_2 = (1, 0, 0) = e_1$ and $v_1 = v_3 = (0, 1, 0) = e_2$. Then, using the notation of the adjugate matrix in (2),

$$\begin{aligned} v_2 \times_B v_3 &= \langle (1, 0, 0), (0, 1, 0)\rangle \operatorname{adj} B \\ &= (0, 0, 1)\begin{pmatrix} \alpha_1 & \beta_3 & \beta_2 \\ \beta_3 & \alpha_2 & \beta_1 \\ \beta_2 & \beta_1 & \alpha_3 \end{pmatrix} \\ &= (\beta_2, \beta_1, \alpha_3) \end{aligned}$$

and hence

$$\begin{aligned} \langle v_1, v_2, v_3\rangle_B &= \langle (0,1,0), (\beta_2, \beta_1, \alpha_3)\rangle \begin{pmatrix} \alpha_1 & \beta_3 & \beta_2 \\ \beta_3 & \alpha_2 & \beta_1 \\ \beta_2 & \beta_1 & \alpha_3 \end{pmatrix} \\ &= (\alpha_3, 0, -\beta_2) \begin{pmatrix} \alpha_1 & \beta_3 & \beta_2 \\ \beta_3 & \alpha_2 & \beta_1 \\ \beta_2 & \beta_1 & \alpha_3 \end{pmatrix} \\ &= \left(\alpha_1\alpha_3 - \beta_2^2, \alpha_3\beta_3 - \beta_1\beta_2, 0\right). \end{aligned}$$

Now use the fact that $\operatorname{adj}(\operatorname{adj} B) = (\det B)\, B$ to obtain

$$\begin{aligned} \operatorname{adj} \begin{pmatrix} \alpha_1 & \beta_3 & \beta_2 \\ \beta_3 & \alpha_2 & \beta_1 \\ \beta_2 & \beta_1 & \alpha_3 \end{pmatrix} &= \begin{pmatrix} \alpha_2\alpha_3 - \beta_1^2 & \beta_1\beta_2 - \alpha_3\beta_3 & \beta_1\beta_3 - \alpha_2\beta_2 \\ \beta_1\beta_2 - \alpha_3\beta_3 & \alpha_1\alpha_3 - \beta_2^2 & \beta_2\beta_3 - \alpha_1\beta_1 \\ \beta_1\beta_3 - \alpha_2\beta_2 & \beta_2\beta_3 - \alpha_1\beta_1 & \alpha_1\alpha_2 - \beta_3^2 \end{pmatrix} \\ &= (\det B) \begin{pmatrix} a_1 & b_3 & b_2 \\ b_3 & a_2 & b_1 \\ b_2 & b_1 & a_3 \end{pmatrix} \end{aligned}$$

so that

$$\langle v_1, v_2, v_3\rangle_B = (\det B)\,(a_2, -b_3, 0).$$

Since by (1) we have $[v_1, v_2]_B = e_1 B e_2^T = b_3$ and $[v_1, v_3]_B = e_2 B e_2^T = a_2$, it follows that

$$\begin{aligned} (\det B)\,(a_2, -b_3, 0) &= (\det B)\left[(v_1 \cdot_B v_3)\, e_1 - (v_1 \cdot_B v_2)\, e_2\right] \\ &= (\det B)\left[(v_1 \cdot_B v_3)\, v_2 - (v_1 \cdot_B v_2)\, v_3\right]. \end{aligned}$$

From this, we deduce that $\lambda = \det B$ and hence

$$\langle v_1, v_2, v_3\rangle_B = (\det B)\left[(v_1 \cdot_B v_3)\, v_2 - (v_1 \cdot_B v_2)\, v_3\right]$$

as required. ∎

The B-vector product itself is not an associative operation, but by the anti-symmetric property of B-vector products, we see that

$$\langle v_1, v_2, v_3\rangle_B = -\langle v_1, v_3, v_2\rangle_B.$$

The following result, attributed in the Euclidean case to Jacobi [16], connects the theory of B-vector products to the theory of Lie algebras and links the three B-vector triple products which differ by an even permutation of the indices, hence generalising Jacobi's result.

Theorem 6 (Generalised Jacobi identity) *For vectors v_1, v_2 and v_3 in the inner product space $\mathbb{F}^3$ equipped with the B-scalar product, we have*

$$\langle v_1, v_2, v_3\rangle_B + \langle v_2, v_3, v_1\rangle_B + \langle v_3, v_1, v_2\rangle_B = \mathbf{0}.$$

Proof. Apply the Generalised Lagrange's formula to each of the three summands to get

$$\langle v_1, v_2, v_3 \rangle_B = (\det B) \left[[v_1, v_3]_B \, v_2 - [v_1, v_2]_B \, v_3 \right]$$

as well as

$$\langle v_2, v_3, v_1 \rangle_B = (\det B) \left[[v_1, v_2]_B \, v_3 - [v_2, v_3]_B \, v_1 \right]$$

and

$$\langle v_3, v_1, v_2 \rangle_B = (\det B) \left[[v_2, v_3]_B \, v_1 - [v_1, v_3]_B \, v_2 \right].$$

Add the three summands to get the required result. ■

3.3 Scalar quadruple products over general inner product spaces

Recall that the **Euclidean scalar quadruple product** of vectors v_1, v_2, v_3 and v_4 in the inner product space $\mathbb{F}^3$ equipped with the usual Euclidean scalar product (from [11, pp. 75-76]) is defined as

$$[v_1, v_2; v_3, v_4] = [\langle v_1, v_2 \rangle, \langle v_3, v_4 \rangle].$$

We will similarly define in the inner product space $\mathbb{F}^3$ equipped with the B-scalar product the B**-scalar quadruple product** to be the quantity

$$[v_1, v_2; v_3, v_4]_B = [\langle v_1, v_2 \rangle_B, \langle v_3, v_4 \rangle_B]_B.$$

The following result, which originated from separate works of Binet [2] and Cauchy [4] in the case of the Euclidean scalar quadruple products and also highlighted in [3] and [24, p. 29], allows us to compute B-scalar quadruple products purely in terms of B-scalar products.

Theorem 7 (Generalised Binet-Cauchy identity) *For vectors v_1, v_2, v_3 and v_4 in the inner product space $\mathbb{F}^3$ equipped with the B-scalar product, we have*

$$[v_1, v_2; v_3, v_4]_B = (\det B) \left[[v_1, v_3]_B [v_2, v_4]_B - [v_1, v_4]_B [v_2, v_3]_B \right].$$

Proof. Let $w = \langle v_1, v_2 \rangle_B$, so that by the Generalised calar triple product theorem and Corollary 4,

$$[v_1, v_2; v_3, v_4]_B = [w, v_3, v_4]_B = [v_4, w, v_3]_B.$$

By the Generalised Lagrange formula,

$$\langle w, v_3 \rangle_B = -\langle v_3, v_1, v_2 \rangle_B = (\det B) \left[[v_1, v_3]_B \, v_2 - [v_2, v_3]_B \, v_1 \right]$$

and hence

$$\begin{aligned} [v_1, v_2; v_3, v_4]_B &= \left((\det B) \left[[v_1, v_3]_B \, v_2 - [v_2, v_3]_B \, v_1 \right] \right) \cdot v_4 \\ &= (\det B) \left[[v_1, v_3]_B [v_2, v_4]_B - [v_1, v_4]_B [v_2, v_3]_B \right] \end{aligned}$$

as required. ■

An important special case of the Generalised Binet-Cauchy identity is another result of Lagrange [19], which we now generalise to a general inner product space. We distinguish this from the Generalised Lagrange formula, which computes the B-vector triple product of three vectors, by calling it the Generalised Lagrange identity.

Theorem 8 (Lagrange's identity) *Given vectors v_1 and v_2 in the inner product space $\mathbb{F}^3$ equipped with the B-scalar product, we have*

$$Q_B\left(\langle v_1, v_2\rangle_B\right) = (\det B)\left[Q_B(v_1)\,Q_B(v_2) - [v_1, v_2]_B^2\right].$$

Proof. This immediately follows from the Generalised Binet-Cauchy identity by setting $v_1 = v_3$ and $v_2 = v_4$. ■

Here is another consequence of the Generalised Binet-Cauchy identity, which is somewhat similar to the Generalised Jacobi identity for B-vector triple products.

Corollary 9 *For vectors v_1, v_2, v_3 and v_4 in the inner product space $\mathbb{F}^3$ equipped with the B-scalar product, we have*

$$[v_1, v_2; v_3, v_4]_B + [v_2, v_3; v_1, v_4]_B + [v_3, v_1; v_2, v_4]_B = 0.$$

Proof. From the Generalised Binet-Cauchy identity, the three summands evaluate to

$$[v_1, v_2; v_3, v_4]_B = (\det B)\left[[v_1, v_3]_B\,[v_2, v_4]_B - [v_1, v_4]_B\,[v_2, v_3]_B\right]$$

as well as

$$[v_2, v_3; v_1, v_4]_B = (\det B)\left[[v_2, v_1]_B\,[v_3, v_4]_B - [v_2, v_4]_B\,[v_3, v_1]_B\right]$$

and

$$[v_3, v_1; v_2, v_4]_B = (\det B)\left[[v_3, v_2]_B\,[v_1, v_4]_B - [v_3, v_4]_B\,[v_1, v_2]_B\right].$$

Add the three summands to obtain the required result. ■

3.4 Vector quadruple products over general inner product spaces

Recall that the **Euclidean vector quadruple product** of vectors v_1, v_2, v_3 and v_4 in the inner product space $\mathbb{F}^3$ equipped with the usual Euclidean scalar product (from [11, pp. 76-77]) is the vector

$$\langle v_1, v_2; v_3, v_4\rangle = \langle\langle v_1, v_2\rangle, \langle v_3, v_4\rangle\rangle.$$

Define similarly the **B-vector quadruple product** in the inner product space $\mathbb{F}^3$ equipped with the B-scalar product as

$$\langle v_1, v_2; v_3, v_4\rangle_B = \langle\langle v_1, v_2\rangle_B, \langle v_3, v_4\rangle_B\rangle_B.$$

The key result here, generalising [11, p. 77] for a general inner product space, is given below.

Theorem 10 (Generalised vector quadruple product theorem) *For vectors v_1, v_2, v_3 and v_4*

in the inner product space $\mathbb{F}^3$ *equipped with the* B*-scalar product, we have*

$$\begin{aligned}\langle v_1, v_2; v_3, v_4\rangle_B &= (\det B)\left([v_1, v_2, v_4]_B\, v_3 - [v_1, v_2, v_3]_B\, v_4\right) \\ &= (\det B)\left([v_1, v_3, v_4]_B\, v_2 - [v_2, v_3, v_4]_B\, v_1\right).\end{aligned}$$

Proof. If $u = \langle v_1, v_2\rangle_B$, then use the Generalised Lagrange formula to get

$$\langle v_1, v_2; v_3, v_4\rangle_B = \langle u, v_3, v_4\rangle_B = (\det B)\left[[u, v_4]_B\, v_3 - [u, v_3]_B\, v_4\right].$$

From the Generalised scalar triple product theorem,

$$[u, v_3]_B = [v_1, v_2, v_3]_B \quad \text{and} \quad [u, v_4]_B = [v_1, v_2, v_4]_B\,.$$

Therefore,

$$\langle v_1, v_2; v_3, v_4\rangle_B = (\det B)\left[[v_1, v_2, v_4]_B\, v_3 - [v_1, v_2, v_3]_B\, v_4\right].$$

Since

$$\langle v_1, v_2; v_3, v_4\rangle_B = -\langle v_3, v_4; v_1, v_2\rangle_B = (v_3 \times_B v_4) \times_B (v_2 \times_B v_1)$$

Corollary 4 gives us

$$\begin{aligned}\langle v_1, v_2; v_3, v_4\rangle_B &= (\det B)\left([v_3, v_4, v_1]_B\, v_2 - [v_3, v_4, v_2]_B\, v_1\right) \\ &= (\det B)\left([v_1, v_3, v_4]_B\, v_2 - [v_2, v_3, v_4]_B\, v_1\right)\end{aligned}$$

which completes the proof. ■

As a corollary, we find a relation satisfied by any four vectors in three-dimensional vector space, extending the result in [11, p. 76] to a general inner product space.

Corollary 11 *Suppose that* v_1*,* v_2*,* v_3 *and* v_4 *are vectors in the inner product space* $\mathbb{F}^3$ *equipped with the* B*-scalar product. Then,*

$$[v_2, v_3, v_4]_B\, v_1 - [v_1, v_3, v_4]_B\, v_2 + [v_1, v_2, v_4]_B\, v_3 - [v_1, v_2, v_3]_B\, v_4 = \mathbf{0}.$$

Proof. This is an immediate consequence of equating the two formulations of the Generalised vector quadruple product theorem, after cancelling the non-zero factor $\det B$. ■

The following result, stated and proven in [21, p. 109] for an inner product space equipped with the Euclidean scalar product, allows us to compute the product of two B-scalar triple products. The proof will rely on the above corollary.

Theorem 12 *Take six vectors* v_1*,* v_2*,* v_3*,* w_1*,* w_2 *and* w_3 *in the inner product space* $\mathbb{F}^3$ *equipped with the* B*-scalar product, and let*

$$G_B = \begin{pmatrix} [v_1, w_1]_B & [v_1, w_2]_B & [v_1, w_3]_B \\ [v_2, w_1]_B & [v_2, w_2]_B & [v_2, w_3]_B \\ [v_3, w_1]_B & [v_3, w_2]_B & [v_3, w_3]_B \end{pmatrix}.$$

Then,

$$[v_1, v_2, v_3]_B \, [w_1, w_2, w_3]_B = (\det B) \, (\det G_B) \, .$$

Proof. Let $a = \langle w_1, w_2 \rangle_B$, so that by applying Corollary 11 on the vectors v_1, v_2, v_3 and a,

$$[v_2, v_3, a]_B \, v_1 - [v_1, v_3, a]_B \, v_2 + [v_1, v_2, a]_B \, v_3 - [v_1, v_2, v_3]_B \, a = \mathbf{0}. \qquad (7)$$

From Corollary 4 and the Generalised Binet-Cauchy identity, we get

$$[v_2, v_3, a]_B = [w_1, w_2; v_2, v_3]_B = (\det B) \, [[w_1, v_2]_B \, [w_2, v_3]_B - [w_1, v_3]_B \, [w_2, v_2]_B]$$

and similarly

$$[v_1, v_3, a]_B = (\det B) \, [[w_1, v_1]_B \, [w_2, v_3]_B - [w_1, v_3]_B \, [w_2, v_1]_B]$$

and

$$[v_1, v_2, a]_B = (\det B) \, [[w_1, v_1]_B \, [w_2, v_2]_B - [w_1, v_2]_B \, [w_2, v_2]_B] \, .$$

Rewriting (7) as

$$[v_1, v_2, v_3]_B \, a = [v_2, v_3, a]_B \, v_1 - [v_1, v_3, a]_B \, v_2 + [v_1, v_2, a]_B \, v_3$$

substitute the above three computed quantities to get

$$\begin{aligned} [v_1, v_2, v_3]_B \, a \;\; = \;\; & ((\det B) \, [[w_1, v_2]_B \, [w_2, v_3]_B - [w_1, v_3]_B \, [w_2, v_2]_B]) \, v_1 \\ & - ((\det B) \, [[w_1, v_1]_B \, [w_2, v_3]_B - [w_1, v_3]_B \, [w_2, v_1]_B]) \, v_2 \\ & + ((\det B) \, [[w_1, v_1]_B \, [w_2, v_2]_B - [w_1, v_2]_B \, [w_2, v_2]_B]) \, v_3 \end{aligned}$$

and take the dot product of each side with w_3, using Corollary 4 again, to get

$$\begin{aligned} [[v_1, v_2, v_3]_B \, a, w_3]_B &= [v_1, v_2, v_3]_B \, [w_1, w_2, w_3]_B \\ &= (\det B) \begin{pmatrix} [w_1, v_2]_B \, [w_2, v_3]_B \, [w_3, v_1]_B - [w_1, v_3]_B \, [w_2, v_2]_B \, [w_3, v_1]_B \\ - [w_1, v_1]_B \, [w_2, v_3]_B \, [w_3, v_2]_B + [w_1, v_3]_B \, [w_2, v_1]_B \, [w_3, v_2]_B \\ + [w_1, v_1]_B \, [w_2, v_2]_B \, [w_3, v_3]_B - [w_1, v_2]_B \, [w_2, v_2]_B \, [w_3, v_3]_B \end{pmatrix} \\ &= (\det B) \begin{vmatrix} [v_1, w_1]_B & [v_1, w_2]_B & [v_1, w_3]_B \\ [v_2, w_1]_B & [v_2, w_2]_B & [v_2, w_3]_B \\ [v_3, w_1]_B & [v_3, w_2]_B & [v_3, w_3]_B \end{vmatrix} \\ &= (\det B) \, (\det G_B) \end{aligned}$$

as required. ∎

In the proof of the above theorem, we may write $G_B = MBN$ where

$$M = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \quad \text{and} \quad N = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}$$

so that we have

$$[v_1, v_2, v_3]_B \, [w_1, w_2, w_3]_B = (\det B)^2 \, (\det M) \, (\det N) \, .$$

As another consequence, we get an expression for the meet of two distinct two-dimensional subspaces.

Corollary 13 *If $U = \text{span}\,(v_1, v_2)$ and $V = \text{span}\,(v_3, v_4)$ are distinct two-dimensional subspaces of $\mathbb{F}^3$ then $\langle v_1, v_2; v_3, v_4 \rangle_B$ spans $U \cap V$.*

Proof. Clearly v is both in U and in V from the Generalised vector quadruple product theorem. We need only show that it is non-zero but this follows from the statement of the aforementioned result, since by assumption v_3 and v_4 are linearly independent and at least one of $[v_1, v_2, v_4]_B$ and $[v_1, v_2, v_3]_B$ must be non-zero since otherwise both v_4 and v_3 lie in U, which contradicts the assumption that the U and V are distinct. ■

A special case occurs when each of the factors of the B-quadruple vector product contains a common vector. This extends the result in [11, p. 80] to a general metrical framework.

Corollary 14 *If v_1, v_2 and v_3 are vectors in the inner product space $\mathbb{F}^3$ equipped with the B-scalar product, then*

$$\langle v_1, v_2; v_1, v_3 \rangle_B = (\det B) \, [v_1, v_2, v_3]_B \, v_1.$$

Proof. This follows from

$$\langle v_1, v_2; v_1, v_3 \rangle_B = (\det B) \left([v_1, v_2, v_3]_B \, v_1 - [v_1, v_2, v_1]_B \, v_3 \right)$$

together with the fact that $[v_1, v_2, v_1]_B = 0$. ■

Yet another consequence is given below, which was alluded to [11, p. 86] in for the Euclidean case.

Theorem 15 *For vectors v_1,v_2 and v_3 in the inner product space $\mathbb{F}^3$ equipped with the B-scalar product,*

$$\left[\langle v_2, v_3 \rangle_B \, , \langle v_3, v_1 \rangle_B \, , \langle v_1, v_2 \rangle_B \right]_B = (\det B) \left([v_1, v_2, v_3]_B \right)^2 .$$

Proof. From Corollary 14,

$$\begin{aligned} \left[\langle v_2, v_3 \rangle_B \, , \langle v_3, v_1 \rangle_B \, , \langle v_1, v_2 \rangle_B \right]_B &= -\left[\langle v_2, v_3 \rangle_B \, , \langle \langle v_1, v_3 \rangle_B \, , \langle v_1, v_2 \rangle_B \rangle_B \right] \\ &= -\left[\langle v_2, v_3 \rangle_B \, , (\det B) \, [v_1, v_3, v_2]_B \, v_1 \right] \\ &= (\det B) \left([v_1, v_2, v_3]_B \right)^2 \end{aligned}$$

as required. ■

It follows that if v_1, v_2 and v_3 are linearly independent, then so are $\langle v_1, v_2 \rangle_B$, $\langle v_2, v_3 \rangle_B$ and $\langle v_3, v_1 \rangle_B$. This also suggests that there is a kind of duality here, which we can clarify by the following result, which is a generalization of Exercise 8 of [21, p. 116] to a general inner product space; it will contain four parts.

Theorem 16 *Suppose that v_1, v_2 and v_3 are linearly independent vectors in $\mathbb{V}^3$, so that $[v_1, v_2, v_3]_B$ is non-zero. Define*

$$w_1 = \frac{\langle v_2, v_3 \rangle_B}{[v_1, v_2, v_3]_B}, \quad w_2 = \frac{\langle v_3, v_1 \rangle_B}{[v_1, v_2, v_3]_B} \quad \text{and} \quad w_3 = \frac{\langle v_1, v_2 \rangle_B}{[v_1, v_2, v_3]_B}.$$

Then, we have that

a)

$$\langle v_1, w_1 \rangle_B + \langle v_2, w_2 \rangle_B + \langle v_3, w_3 \rangle_B = \mathbf{0}$$

b)

$$[v_1, w_1]_B + [v_2, w_2]_B + [v_3, w_3]_B = 3$$

c)

$$[v_1, v_2, v_3]_B \, [w_1, w_2, w_3]_B = \det B$$

d)

$$v_1 = \frac{\langle w_2, w_3 \rangle_B}{[w_1, w_2, w_3]_B}, \quad v_2 = \frac{\langle w_3, w_1 \rangle_B}{[w_1, w_2, w_3]_B} \quad \text{and} \quad v_3 = \frac{\langle w_1, w_2 \rangle_B}{[w_1, w_2, w_3]_B}.$$

Proof. By the Generalised Jacobi identity,

$$\begin{aligned} & \langle v_1, w_1 \rangle_B + \langle v_2, w_2 \rangle_B + \langle v_3, w_3 \rangle_B \\ = & \frac{1}{[v_1, v_2, v_3]_B} \left(\langle v_1, v_2, v_3 \rangle_B + \langle v_2, v_3, v_1 \rangle_B + \langle v_3, v_1, v_2 \rangle_B \right) \\ = & \mathbf{0}. \end{aligned}$$

Moreover, we use the definition of the B-scalar triple product to obtain

$$\begin{aligned} & [v_1, w_1]_B + [v_2, w_2]_B + [v_3, w_3]_B \\ = & \frac{1}{[v_1, v_2, v_3]_B} \left([v_1, v_2, v_3]_B + [v_2, v_3, v_1]_B + [v_3, v_1, v_2]_B \right) \\ = & 3. \end{aligned}$$

By Theorem 15,

$$\begin{aligned} [w_1, w_2, w_3]_B &= \left[\frac{\langle v_2, v_3 \rangle_B}{[v_1, v_2, v_3]_B}, \frac{\langle v_3, v_1 \rangle_B}{[v_1, v_2, v_3]_B}, \frac{\langle v_1, v_2 \rangle_B}{[v_1, v_2, v_3]_B} \right]_B \\ &= \frac{1}{[v_1, v_2, v_3]_B^3} \left[\langle v_2, v_3 \rangle_B, \langle v_3, v_1 \rangle_B, \langle v_1, v_2 \rangle_B \right]_B \\ &= \frac{1}{[v_1, v_2, v_3]_B^3} (\det B) \left([v_1, v_2, v_3]_B \right)^2 \\ &= \frac{\det B}{[v_1, v_2, v_3]_B} \end{aligned}$$

and hence

$$[v_1, v_2, v_3]_B \, [w_1, w_2, w_3]_B = \det B.$$

Given this result, we then have

$$\begin{aligned}\langle w_2, w_3\rangle_B &= \left\langle \frac{\langle v_3, v_1\rangle_B}{[v_1, v_2, v_3]_B}, \frac{\langle v_1, v_2\rangle_B}{[v_1, v_2, v_3]_B} \right\rangle_B \\ &= -\frac{1}{[v_1, v_2, v_3]_B^2} \langle v_1, v_3; v_1, v_2\rangle_B\end{aligned}$$

which by Corollary 13 becomes

$$\langle w_2, w_3\rangle_B = -\frac{1}{[v_1, v_2, v_3]_B^2} (\det B) [v_1, v_3, v_2]_B \, v_1.$$

By the Generalised scalar triple product theorem,

$$\langle w_2, w_3\rangle_B = \frac{(\det B)}{[v_1, v_2, v_3]_B} v_1$$

and by the previous result in this theorem, which has been already proven,

$$\langle w_2, w_3\rangle_B = [w_1, w_2, w_3]_B \, v_1.$$

Thus

$$v_1 = \frac{\langle w_2, w_3\rangle_B}{[w_1, w_2, w_3]_B}$$

and the results for v_2 and v_3 are similar. ■

Parts a) and d) of this result also proven in [11, p. 86] for an inner product space equipped with the Euclidean scalar product; furthermore, we see that part c) of this result is a special case of Theorem 12, where the vectors w_1, w_2 and w_3 are dependent on v_1, v_2 and v_3 through the use of B-vector products.

4 Scope of applications

The use of vector products has been prevalent in understanding more advanced ideas in geometry, notably in relativistic geometry. Here, a **Minkowski scalar product** [20] in $\mathbb{F}^3$, given for vectors $v = (v_1, v_2, v_3)$ and $w = (w_1, w_2, w_3)$ by

$$v \cdot w = v_1 w_1 + v_2 w_2 - v_3 w_3$$

can be represented by the symmetric bilinear form with matrix representation

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Euclidean geometry and relativistic geometry, among many other non-Euclidean geometries, can be unified through Klein's Erlangen program (see [17] and [18]) by organising geometry in terms of projective geometry, a less restrictive form of affine geometry, over a general symmetric bilinear form as described above. Some form of this treatment is also given in [22] and [23].

Coordinate-free approaches may also exist, which can generalise the ideas of the vector product over a general inner product space to arbitrary dimensions. This may be explored in a future paper by the author, where the works of [8] and [15] in the field of geometric algebra will be crucial in such a setup. The vector product is closely linked to the idea of Hodge star operators in geometric algebra; seen in [9, p. 15], this study would be a desirable first step in generalising the vector product over a general inner product space to coordinate-free approaches.

5 Acknowledgment

This research was conducted during the author's doctoral candidature at UNSW Sydney in Sydney, NSW, Australia.

References

[1] Anton, H. & Rorres, C.: *Elementary Linear Algebra with Applications* (9th ed.). John Wiley & Sons, Inc., Hoboken, NJ (2005).

[2] Binet, J. F. M.: Mémoire sur un systeme de formules analytiques, et leur application à des considerations géométriques. *J. Ec. Polytech.*, 9, 280-302 (1812).

[3] Brualdi, R. A. & Schneider, H.: Determinantal identities: Gauss, Schur, Cauchy, Sylvester, Kronecker, Jacobi, Binet, Laplace, Muir, and Cayley. *Linear Algebra Appl.*, 52-53, 769-791 (1983).

[4] Cauchy, A.: Memoire sur les fonctions qui ne peuvent obtenir que deux valeurs égales et des signes contraires par suite des transpositions opérées entre les variables qu'elles renferment. *J. Ec. Polytech.*, 17, 29-112 (1815).

[5] Chapman, S. & Milne, E. A.: The proof of the formula for the vector triple product. *Math. Gaz.*, 23(253), 35-38 (1939).

[6] Clifford, W. K.: *Elements of Dynamic.* MacMillan & Co., London, UK (1878).

[7] Collomb, C.: *A tutorial on inverting 3 by 3 matrices with cross products.* http://www.emptyloop.com/technotes (n.d.). Accessed 15 August 2021.

[8] Doran, C. J. L. & Lasenby, A. N.: *Geometric Algebra for Physicists.* Cambridge University Press, Cambridge, UK (2003).

[9] Flanders, H.: *Differential Forms with Applications to the Physical Sciences.* Academic Press, Inc., New York, NY (1963).

[10] Gantmacher, F. R.: *The Theory of Matrices* (Vol. 1). Chelsea Publishing Company, New York, NY (1960).

[11] Gibbs, J. W. & Wilson, E. B.: *Vector Analysis: A text-book for the use of students of mathematics and physics founded upon the lectures of J. Willard Gibbs*. Yale University Press, New Haven, CT (1901).

[12] Hamilton, W.R.: *Lectures on Quaternions*. Hodges and Smith, Dublin, Ireland (1853).

[13] Havlicek, H. & Weiß, G.: Altitudes of a Tetrahedron and Traceless Quadratic Forms. *Amer. Math. Monthly*, 110(8), 679-693 (2003).

[14] Heaviside, O.: *Electromagnetic Theory* (Vol. 1). The Electrician Printing and Publishing Co., London, UK (1893).

[15] Hestenes, D. & Sobczyk, G.: *Clifford Algebra to Geometric Calculus, a Unified Language for Mathematics and Physics*. D. Reidel Publishing Company, Dordrecht, Netherlands (1984).

[16] Jacobi, C. G. J.: *Fundamenta Nova Theoriae Functionum Ellipticarum*. Pontrieu & Co. and Treuttel & Wuerz, Paris, France (1829).

[17] Klein, F.: Vergleichende Betrachtungen über neuere geometrische Forschungen. *Math. Ann.*, 43, 63-100 (1872).

[18] Klein, F.: *Elementary Mathematics from an Advanced Standpoint: Geometry*. Dover, New York, NY (2004).

[19] Lagrange, J. L.: Solutions analytiques de quelques problèmes sur les pyramides triangulaires. *Oeuvres de Lagrange*, 3, 661-692 (1773).

[20] Minkowski, H.: Das Relativitätsprinzip. *Ann. Phys.*, 352(15), 927-938 (1907).

[21] Narayan, S.: *A Textbook of Vector Analysis*. S. Chand & Company Ltd, New Delhi, India (1961).

[22] Schröder, E. M.: Metric Geometry. In F. Buekenhout (Ed.). *Handbook of Incidence Geometry: Buildings and Foundations* (945-1013). North-Holland Publishing Co., Amsterdam, Netherlands (1995).

[23] Snapper, E. & Troyer, R. J.: *Metric Affine Geometry*. Academic Press, New York, NY (1971).

[24] Spiegel, M. R.: *Schaum's Outline of Vector Analysis*. McGraw-Hill Inc, New York, NY (1959).

[25] Strang, G.: *Linear Algebra and its Applications* (3rd ed.). Harcourt Brace Jovanovich, San Diego, CA (1988).